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Topology of superfluid ^{3}He in cylindrical pores

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Abstract. The effect of boundary conditions on the topological classification of line singularities in a cylindrical pore is studied for the A, B and A_1 phases of superfluid ^{3}He.

1. Introduction

Topological homotopy group methods have been applied to the classification of point and line singularities in the A and B phases of superfluid ^{3}He by Toulouse and Kléman (1976), Volovik and Mineev (1977a, b) and Cross and Brinkman (1977), and to the A_1 phase by Bailin and Love (1978). These methods allow one to discover which textures of the superfluid can deform continuously into each other, and which are forbidden to decay into one another by a topological conservation law. However, when the superfluid is in a container, the topology is further restricted by the boundary conditions at the walls, and textures may become topologically inequivalent which were equivalent in an open system. This is because even if two textures are equivalent in the open system and are both compatible with the boundary conditions, they may be unable to deform into each other without violating the boundary conditions at some intermediate stage. More formally, the constraints imposed by the boundary conditions result in the topological space of allowed order parameters being 'smaller' on the boundary than in the interior of the superfluid, with consequently more topological quantum numbers.

Two questions then arise. First, how large are the free energy barriers associated with these new topological quantum numbers? In the open system, ignoring the effect of dipolar interactions, it is necessary to leave the phase altogether to evade the conditions imposed by the topological conservation laws, and so there is a free energy barrier of the order of the bulk free energy. To escape the constraints imposed by the new topological quantum numbers produced by the boundary conditions, it is necessary to violate the boundary conditions, and for the A and A_1 phases this involves a free energy of order $\xi_0 F_B/R$, where F_B is the total bulk free energy of the sample of superfluid, R is the radius of the pore, and $\xi_0 \simeq 10^{-6}$ cm (see, for example, Leggett 1975). On the other hand, the total bending free energy of the superfluid is of order $\xi_0^2 F_B/R^2$, and so in pores much *wider* than 10^{-6} cm the free energy associated with the boundary conditions is much larger than the total bending free energy. Consequently, the topological quantum numbers associated with the boundary conditions will be 'good quantum numbers'. For the B phase the surface energy is much smaller, and the

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topological quantum numbers associated with the boundary conditions are only 'good' in pores *wider* than 1 cm.

The second question that arises is how significant the new topological quantum numbers are for the texture of the superfluid as a whole. One might imagine that two textures which are equivalent in an open system but inequivalent in the pore might deform into each other in the pore in all but a narrow layer of thickness d round the boundary. If this thickness were very small, the new topological quantum numbers would be of limited significance. This healing distance d can be estimated by observing that it will not be energetically favourable for rapid healing at the boundary to occur if the bending free energy involved is larger than the typical difference in bending free energy between two textures. Thus, we expect $d/R \gtrsim [\ln(R/\xi_0)]^{-1}$ where R is the radius of the pore and $\xi_0 \simeq 10^{-6}$ cm. Consequently, for pores of radius *less* than 1 cm, healing will occur over a significant fraction of the pore, and the effect of the boundary conditions on the topology will have an important effect on the texture as a whole. Experiments with one-micrometre pores (Saunders *et al* 1978) should therefore exhibit topological quantum numbers associated with the boundary conditions.

In the present paper, we study the homotopy groups associated with the order parameter space of the superfluid at the boundary, and the corresponding topological quantum numbers. The A, B and A_1 phases are studied both in pores of radius much larger than the dipolar length ($\simeq 10^{-3}$ cm), but less than about 1 cm, and in pores of radius much smaller than the dipolar length, but large compared with 10^{-6} cm. The effect of a magnetic field along the axis of the pore is also considered. It should be emphasised that we concentrate on textures with singularities in the interior of the superfluid, and *not* textures with singularities on the surface. This is in contrast to Mermin (1977) who focuses his attention on surface singularities in the A phase.

The notations we adopt for the order parameters in the A, B and A_1 phases are as follows. In the A phase, the order parameter is of the form

$$A_{\mu i} = \Delta d_\mu \Delta_i, \tag{1.1}$$

where Δ is the magnitude of the order parameter,

$$\boldsymbol{\Delta} = \frac{1}{\sqrt{2}}(\boldsymbol{\alpha}_1 + \mathrm{i}\boldsymbol{\alpha}_2) \tag{1.2}$$

is a vector in ordinary space with $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ real mutually orthogonal unit vectors, and $\boldsymbol{d}$ is a real unit vector in spin space. The $\boldsymbol{l}$ vector is given by

$$\boldsymbol{l} = \boldsymbol{\alpha}_1 \times \boldsymbol{\alpha}_2. \tag{1.3}$$

In pores of radius large compared with the dipolar length, the dipolar interactions impose the constraint $\boldsymbol{d} = \pm \boldsymbol{l}$. A magnetic field aligns $\boldsymbol{d}$ perpendicular to it.

In the B phase the order parameter is of the form

$$A_{\mu i} = \Delta \, \mathrm{e}^{\mathrm{i}\chi} R_{\mu i}, \tag{1.4}$$

where Δ is the magnitude of the order parameter, χ is the phase, and $R_{\mu i}$ is a rotation matrix corresponding to rotation through an arbitrary angle α about an arbitrary direction $\boldsymbol{n}$. If the pore is large on the scale of the dipolar length, α is constrained to be $\cos^{-1}(-\frac{1}{4})$. Finally, in the A_1 phase the order parameter is of the form

$$A_{\mu i} = \Delta d_\mu \Delta_i, \tag{1.5}$$

where Δ is the magnitude of the order parameter,

$$\mathbf{\Delta} = \frac{1}{\sqrt{2}}(\boldsymbol{\alpha}_1 + i\boldsymbol{\alpha}_2) \tag{1.6}$$

is a vector in ordinary space with $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ real mutually orthogonal unit vectors (as for the A phase), but now

$$\boldsymbol{d} = \frac{1}{\sqrt{2}}(\boldsymbol{\beta}_1 + i\boldsymbol{\beta}_2) \tag{1.7}$$

is a spin-space vector with $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ real mutually orthogonal unit vectors. In pores of large radius compared with the dipolar length, the dipolar interactions constrain $\boldsymbol{l} = \boldsymbol{\alpha}_1 \times \boldsymbol{\alpha}_2$ and $\boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2$ to be at right angles to each other. A magnetic field aligns $\boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2$ parallel to it.

2. A-phase topology

In this section, we consider the topological space of allowed order parameters for the superfluid near the boundary of a cylindrical pore, beginning with the case where the pore is of radius small compared with the dipolar length, so that the effect of dipolar interactions can be neglected.

In the interior of the superfluid the space of order parameters is

$$R = (SO_3 \times S^2)/Z_2 \tag{2.1}$$

and the classification of line singularities is given by the first homotopy group

$$\pi_1(R) = Z_4 \tag{2.2}$$

(see, for example, Volovik and Mineev 1977a, b). On the boundary, only $\boldsymbol{l} = \pm\hat{\boldsymbol{\rho}}$ is allowed, where $\hat{\boldsymbol{\rho}}$ is the unit radius vector in cylindrical coordinates, and the restricted order parameter space is (for $\boldsymbol{l} = +\hat{\boldsymbol{\rho}}$, say)

$$\tilde{R} = (S^1 \times S^2)/Z_2, \tag{2.3}$$

where the S^1 is the group $\{R(\alpha\hat{\boldsymbol{z}})\}$ defined by writing the general orbital order parameter $\mathbf{\Delta}$ as

$$\mathbf{\Delta} = R(\tfrac{1}{2}\pi\hat{\boldsymbol{\phi}})R(\alpha\hat{\boldsymbol{z}})\frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}}), \tag{2.4}$$

where $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$ are fixed orthonormal vectors, and $\hat{\boldsymbol{\phi}}$ is the unit azimuthal vector in cylindrical coordinates. The S^2 is associated with the unit vector $\boldsymbol{d}$, as before. The left cosets with respect to Z_2 are formed as usual to take account of the possibility of multiplying the orbital and spin parts of the order parameter separately by -1 without changing the order parameter. (We use $R(\alpha\hat{\boldsymbol{n}})$ to denote a rotation through α about the axis $\hat{\boldsymbol{n}}$.)

The homotopy group $\pi_1(\tilde{R})$ can be calculated using an exact sequence of homomorphisms (see, for example, Steenrod 1951, Volovik and Mineev 1977a, b). Any ambiguity which arises can be resolved by studying the properties of the natural mapping from $S^1 \times S^2$ to $(S^1 \times S^2)/Z_2$ (see the appendix). The result is

$$\pi_1(\tilde{R}) = Z. \tag{2.5}$$

For example, the texture

$$\begin{aligned}\sqrt{2}\mathbf{\Delta} &= (\hat{\boldsymbol{\phi}} + \mathrm{i}\hat{\boldsymbol{z}})\exp[\mathrm{i}(m+\tfrac{1}{2})\phi] \\ \boldsymbol{d} &= \hat{\boldsymbol{x}}\cos[(n+\tfrac{1}{2})\phi] + \hat{\boldsymbol{y}}\sin[(n+\tfrac{1}{2})\phi]\end{aligned} \tag{2.6}$$

may be assigned topological quantum numbers $N(Z) = 2m - 1$. On the other hand, in the Z_4 classification appropriate to an open system it has $N(Z_4) = (2m-1) \pmod 4$. Thus, for example, the textures with $m = 0$ and $m = 2$ are equivalent in the absence of boundary conditions, but become inequivalent in a pore.

(In addition to the (first) homotopy group classification given above, we must of course also treat $\boldsymbol{l} = \hat{\boldsymbol{\rho}}$ and $\boldsymbol{l} = -\hat{\boldsymbol{\rho}}$ as topologically inequivalent since there is no way of proceeding from one configuration to the other without violating the boundary conditions at an intermediate stage. In other words, the manifold of *all* allowed order parameters, including $\boldsymbol{l} = \pm\hat{\boldsymbol{\rho}}$, consists of two disconnected pieces.)

If a magnetic field is applied along the axis of the pore, then the $\boldsymbol{d}$ vector is confined to a circle, and the space of order parameters in the interior of the superfluid will be

$$R = (\mathrm{SO}_3 \times S^1)/Z_2, \tag{2.7}$$

leading to homotopy group

$$\pi_1(R) = Z_2 + Z. \tag{2.8}$$

On the boundary of the superfluid, where $\boldsymbol{l} = \pm\hat{\boldsymbol{\rho}}$, the appropriate order parameter space will be (for $\boldsymbol{l} = +\hat{\boldsymbol{\rho}}$ say)

$$\tilde{R} = (S^1 \times S^1)/Z_2, \tag{2.9}$$

where the first S^1 is defined by writing the general form for $\mathbf{\Delta}$ as

$$R(\tfrac{1}{2}\pi\hat{\boldsymbol{\phi}})R(\alpha\hat{\boldsymbol{z}})\frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}}), \tag{2.10}$$

and the second S^1 is associated with $\boldsymbol{d}$. The corresponding homotopy group is

$$\pi_1(\tilde{R}) = Z + Z. \tag{2.11}$$

For example, the texture

$$\begin{aligned}\sqrt{2}\mathbf{\Delta} &= (\hat{\boldsymbol{\phi}} + \mathrm{i}\hat{\boldsymbol{z}})\exp[\mathrm{i}(m+\tfrac{1}{2})\phi] \\ \boldsymbol{d} &= \hat{\boldsymbol{x}}\cos(n+\tfrac{1}{2})\phi + \hat{\boldsymbol{y}}\sin(n+\tfrac{1}{2})\phi\end{aligned} \tag{2.12}$$

may be assigned topological quantum numbers

$$N_1(Z) = m - n - 1, \qquad N_2(Z) = 2n + 1.$$

In the absence of boundary conditions, it has

$$N(Z_2) = (m - n - 1) \bmod 2, \qquad N(Z) = 2n + 1$$

and, for example, $m = 0$ and $m = 2$ become equivalent. In fact the homotopy group given by (2.11) is correct for the magnetic field in an arbitrary direction, although the specific example (2.12) would need modification.

If the pore radius is greater than the dipolar length, then dipolar interactions result in a smaller order parameter space. In the interior of the superfluid

$$R = SO_3 \tag{2.13}$$

and

$$\pi_1(R) = Z_2, \tag{2.14}$$

(see, for example, Toulouse and Kléman 1976), and on the boundary

$$\tilde{R} = S^1, \tag{2.15}$$

where the S^1 associated with $\mathbf{\Delta}$ is defined as in equation (2.10). We have immediately,

$$\pi_1(\tilde{R}) = Z. \tag{2.16}$$

The texture

$$\sqrt{2}\mathbf{\Delta} = (\hat{\boldsymbol{\phi}} + i\hat{\boldsymbol{z}})\, e^{im\phi}, \qquad \boldsymbol{d} = \pm\hat{\boldsymbol{\rho}} \tag{2.17}$$

may be assigned topological quantum number $N(Z) = m - 1$, whereas in the absence of boundary conditions such textures are only inequivalent modulo 2.

Since $\boldsymbol{d} = \pm\boldsymbol{l}$ is already perpendicular to the axis of the pore, a magnetic field applied parallel to the axis will not impose any further restriction on the space of order parameters. This means that the discussion of the boundary topology given in equations (2.15), (2.16), and (2.17), still applies. This should be compared with the situation with a strong magnetic field along the axis of the pore in the absence of boundary conditions, where

$$R = S^1 \times S^1 \tag{2.18}$$

and

$$\pi_1(R) = Z + Z. \tag{2.19}$$

(Starting from a reference triad $\sqrt{2}\mathbf{\Delta} = \hat{\boldsymbol{y}} + i\hat{\boldsymbol{z}}$ the first S^1 corresponds to rotation of $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ about the x axis, and the second S^1 corresponds to rotation of the $\boldsymbol{l}$ vector about the z axis.) In the absence of boundary conditions the texture of equation (2.17) may be assigned topological quantum numbers

$$N_1(Z) = m, \qquad N_2(Z) = 1.$$

More generally the homotopy group is unaltered for a strong magnetic field in an arbitrary direction, but the specific example (2.17) would need modification. This is because a strong magnetic field $\boldsymbol{H}$ confines $\boldsymbol{d}$ to a plane perpendicular to $\boldsymbol{H}$, and minimisation of the dipolar energy subject to this constraint yields, as before, precisely two distinct values of $\boldsymbol{d}$.

3. B-phase topology

For the B phase the surface free energy is of order $\xi_0 F_D/R$, where F_D is the total dipolar free energy of the sample, R is the pore radius and $\xi_0 \simeq 10^{-6}$ cm (see, for example, Smith *et al* 1977). On the other hand the total bending free energy is of order $\xi_0^2 10^6 F_D/R^2$, and so in pores *narrower* than 1 cm it is larger than the surface free energy. Thus in such pores any constraints on the order parameter arise purely from the bulk,

dipolar and magnetic field contribution to the free energy, and there are no new quantum numbers associated with surface effects. We therefore consider only pores wider than 1 cm. The effect of dipolar interactions then leads to

$$R = S^1 \times S^2 \tag{3.7}$$

and

$$\pi_1(R) = Z. \tag{3.8}$$

(See, for example, Volovik and Mineev (1977a, b).) On the boundary, only $\boldsymbol{n} = \pm\hat{\boldsymbol{\rho}}$ is allowed. The order parameter space appropriate to the boundary is

$$\tilde{R} = S^1 \tag{3.9}$$

and

$$\pi_1(\tilde{R}) = Z, \tag{3.10}$$

where the S^1 is associated with the phase of the order parameter. In this case, the boundary conditions have no effect on which textures are topologically inequivalent, apart from the obvious one associated with $\boldsymbol{n} = \pm\hat{\boldsymbol{\rho}}$ being the only allowed configurations.

The preceding considerations are unaffected by the presence of a large magnetic field in the plane of the wall. This is because although such a field modifies the boundary condition (see Leggett 1975), it still fixes $\boldsymbol{n}$ to have one of two distinct values. Thus the topology is unmodified. (In an *open* system with a magnetic field $R = S^1 \times S^1$ and $\pi_1(R) = Z + Z$ for small volumes; for large volumes $R = S^1$ and $\pi_1(R) = Z$.)

4. A_1-phase topology

The normal experimental situation for the A_1 phase is a large magnetic field so that the phase has a sufficient width in temperature. We take the magnetic field to be along the axis of the pore, and consider first the case where the pore is of small radius compared with the dipolar length, so that dipolar interactions may be neglected. In that case, the order parameter space for the interior of the superfluid is

$$R = (SO_3 \times S^1)/S^1 \tag{4.1}$$

and using an exact sequence of homomorphisms one finds that

$$\pi_1(R) = Z_2. \tag{4.2}$$

(See Steenrod (1951) and Bailin and Love (1978).)

On the boundary only $\boldsymbol{l} = \pm\hat{\boldsymbol{\rho}}$ is allowed, and the space of order parameters is (for $\boldsymbol{l} = +\hat{\boldsymbol{\rho}}$, say)

$$\tilde{R} = (S^1 \times S^1)/S^1, \tag{4.3}$$

where the first S^1 in the numerator is associated with $\boldsymbol{\Delta}$ and can be defined by writing the general allowed $\boldsymbol{\Delta}$ as

$$\boldsymbol{\Delta} = R(\tfrac{1}{2}\pi\hat{\boldsymbol{\phi}})R(\alpha\hat{\boldsymbol{z}})\frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}}), \tag{4.4}$$

and the second S^1 is defined by writing the general allowed $\boldsymbol{d}$ in a magnetic field as

$$\boldsymbol{d} = R(\beta\hat{\boldsymbol{z}})\frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}}).$$

($R(\alpha\hat{\boldsymbol{n}})$ denotes a rotation through α about the axis $\hat{\boldsymbol{n}}$.) The $S^1 \times S^1$ in the numerator is thus homeomorphic to $(R(\alpha\hat{\boldsymbol{z}}), R(\beta\hat{\boldsymbol{z}}))$. The S^1 in the denominator is the subgroup $(R(\gamma\hat{\boldsymbol{z}}), R(-\gamma\hat{\boldsymbol{z}}))$ of this, and corresponds to the freedom to multiply $\boldsymbol{\Delta}$ by $\exp(-\mathrm{i}\gamma)$ and $\boldsymbol{d}$ by $\exp(\mathrm{i}\gamma)$ without changing the order parameter. It follows that the order parameter depends only on $\alpha+\beta$ and that R is just S^1. Thus

$$\pi_1(\tilde{R}) = Z. \tag{4.5}$$

As an example, the texture

$$\sqrt{2}\boldsymbol{\Delta} = \mathrm{e}^{\mathrm{i}m\phi}(\hat{\boldsymbol{\phi}} + \mathrm{i}\hat{\boldsymbol{z}})$$
$$\sqrt{2}\boldsymbol{d} = \hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}} \tag{4.6}$$

can be assigned topological quantum number $N(Z) = m - 1$. This contrasts with the situation without boundary conditions, where all even values of m are topologically equivalent and all odd values of m are equivalent.

If the radius of the pore is large compared with the dipolar length, then the order parameter space for the interior of the superfluid is

$$R = (S^1 \times S^1 \times S^1)/S^1 \tag{4.7}$$

with

$$\pi_1(R) = Z + Z. \tag{4.8}$$

(See Bailin and Love (1978).)

On the boundary, the appropriate order parameter space is

$$\tilde{R} = (S^1 \times S^1)/S^1 \tag{4.9}$$

exactly as for negligible dipolar forces. (On the boundary $\boldsymbol{d}$ and $\boldsymbol{\Delta}$ are automatically in the correct configuration to minimise the dipolar energy when a magnetic field is applied along the axis of the pore.) Accordingly, the texture of equation (4.6) can again be assigned a topological quantum number $N(Z) = m - 1$. This should be compared with the case where the boundary condition is ignored as in equation (4.8), and there are two integer quantum numbers $N_1(Z) = 1$ and $N_2(Z) = m$.

We also discuss the situation when the magnetic field is negligible. To realise this situation experimentally it may be necessary to study the metastable A_1 phase prepared by switching off the magnetic field. Consider first a pore of radius small compared with the dipolar length. In that case, the order parameter space for the interior of the superfluid is

$$R = (\mathrm{SO}_3 \times \mathrm{SO}_3)/S^1 \tag{4.10}$$

and

$$\pi_1(R) = Z_2. \tag{4.11}$$

(See Bailin and Love (1978).)

On the boundary, $\boldsymbol{l} = \pm\hat{\boldsymbol{\rho}}$, and for $\boldsymbol{l} = +\hat{\boldsymbol{\rho}}$

$$\tilde{R} = (S^1 \times \mathrm{SO}_3)/S^1, \tag{4.12}$$

where the S^1 in the numerator is defined by writing the general allowed $\mathbf{\Delta}$ in the form given in equation (4.4), and the SO_3 is associated with $\boldsymbol{d}$ as before. The $S^1 \times SO_3$ in the numerator is thus homeomorphic to $(R(\alpha\hat{\boldsymbol{z}}), R(\beta\hat{\boldsymbol{n}}))$ where $\hat{\boldsymbol{n}}$ is a general axis of rotation, and the S^1 in the denominator is the $(R(\gamma\hat{\boldsymbol{z}}), R(-\gamma\hat{\boldsymbol{z}}))$ subgroup. Using an exact sequence of homomorphisms we find

$$\pi_1(\tilde{R}) = Z_2. \tag{4.13}$$

For example, the texture

$$\sqrt{2}\mathbf{\Delta} = \hat{\boldsymbol{z}} + \mathrm{i}\hat{\boldsymbol{\phi}}, \qquad \sqrt{2}\boldsymbol{d} = \mathrm{e}^{\mathrm{i}m\phi}(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}}) \tag{4.14}$$

may be assigned a topological quantum number

$$N(Z_2) = (m+1) \bmod 2,$$

which is the same classification as for an open system.

If the magnetic field is negligible, and the pore radius is large compared with the dipolar length, then the order parameter space for the interior of the superfluid becomes

$$R = (SO_3 \times S^1 \times S^1)/S^1, \tag{4.15}$$

with homotopy group

$$\pi_1(R) = Z_2 + Z. \tag{4.16}$$

(See Bailin and Love (1978).)

On the boundary, the order parameter space is restricted to

$$\tilde{R} = (S^1 \times S^1 \times S^1)/S^1, \tag{4.17}$$

where the three numerator S_1's can be defined by writing the general allowed $\mathbf{\Delta}$ as

$$\mathbf{\Delta} = R(\tfrac{1}{2}\pi\hat{\boldsymbol{\phi}})R(\alpha\hat{\boldsymbol{z}})\frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}}) \tag{4.18}$$

and the general allowed $\boldsymbol{d}$ as

$$\boldsymbol{d} = R(\tfrac{1}{2}\pi\hat{\boldsymbol{\phi}})R(\gamma\hat{\boldsymbol{z}})R(\beta\hat{\boldsymbol{x}})\frac{1}{\sqrt{2}}(\hat{\boldsymbol{y}} + \mathrm{i}\hat{\boldsymbol{z}}). \tag{4.19}$$

As before, the denominator S^1 corresponds to the freedom to multiply $\mathbf{\Delta}$ by $\mathrm{e}^{-\mathrm{i}\gamma}$ and $\boldsymbol{d}$ by $\mathrm{e}^{\mathrm{i}\gamma}$ without changing the order parameter. It follows that the order parameter depends only on $\alpha + \beta$ and that R is $S^1 \times S^1$. Thus

$$\pi_1(\tilde{R}) = Z + Z. \tag{4.20}$$

As an example the texture

$$\mathbf{\Delta} = \mathrm{e}^{\mathrm{i}m\phi}\frac{1}{\sqrt{2}}(\hat{\boldsymbol{\phi}} + \mathrm{i}\hat{\boldsymbol{z}}) \tag{4.21}$$

$$\sqrt{2}\boldsymbol{d} = \hat{\boldsymbol{\rho}} - \mathrm{i}[\hat{\boldsymbol{\phi}}\cos(n\phi) + \hat{\boldsymbol{z}}\sin(n\phi)] \tag{4.22}$$

may be assigned the topological quantum numbers

$$N_1(Z) = m + n, \qquad N_2(Z) = n + 1.$$

On the other hand, in the absence of boundary conditions, it would be classified according to equation (4.16) with

$$N(Z_2) = (n+1) \bmod 2, \qquad N(Z) = m + n.$$

(The topological quantum numbers in the absence of boundary conditions are being defined as in Bailin and Love (1978). The connection with this paper is most easily made by rewriting $\mathbf{\Delta}$ in the present equation (4.18) as

$$\mathbf{\Delta} = R(\tfrac{1}{2}\pi\hat{\boldsymbol{\phi}})R(\gamma\hat{\boldsymbol{z}})R((\alpha-\gamma)\hat{\boldsymbol{z}})\frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}})$$

so that $R(\frac{1}{2}\pi\hat{\boldsymbol{\phi}})R(\gamma\hat{\boldsymbol{z}})$ can be identified as a common rotation applied to $\mathbf{\Delta}$ and $\boldsymbol{d}$.)

5. Conclusions

When account is taken of the constraints provided by the boundary conditions at the walls of a pore, the topology of the enclosed superfluid is described by a 'smaller' topological space. As a result, textures which would otherwise have been topologically equivalent become inequivalent. Non-trivial examples of this arise in the A and A_1 phases. For example in the A phase in a pore of radius small compared with the dipolar length, the textures are classified by an integer and are inequivalent for all integer values of n, whereas in the absence of boundary conditions textures classified by the same integer modulo 4 are topologically equivalent.

Although we are able to classify any texture satisfying the boundary conditions by an element of the appropriate homotopy group, it should be emphasised that the *group* has less significance in this context than it has for open systems. This is because there is no physically significant way (that we know of) of defining the sum of two textures both satisfying the boundary conditions.

A by-product of our analysis is that we have completely classified all surface singularities of ^{3}He with $\boldsymbol{l}$(or $\boldsymbol{n}$) $= +\hat{\boldsymbol{z}}$ (say) where $\hat{\boldsymbol{z}}$ is the normal to the surface. This is because the form of the order parameter for the dipole free A phase (for example), is given by (2.4) without the rotation $R(\frac{1}{2}\pi\hat{\boldsymbol{\phi}})$. However, for surface singularities the addition of homotopy group elements corresponds to physical combination of singularities.

In the case of superfluid ^{3}He in small spherical containers, it is possible, *a priori*, that the boundary conditions can give rise to point singularities where none existed in the corresponding open system (because of the 'smaller' order parameter space available to deform them away). We have considered all of the cases studied for the cylindrical geometry and find that the order parameter space $\tilde{R}$ is unaltered in spherical geometry. A study of the second homotopy groups $\pi_2(\tilde{R})$ shows that no point singularities arise which were not already present in the open system. And, of course, the restriction $\boldsymbol{l}$ or $\boldsymbol{n} = \pm\hat{\boldsymbol{r}}$ eliminates some of the point singularities which occurred in the absence of boundary conditions.

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Appendix

For the A phase in small volumes the order parameter space on the boundary is

$$\tilde{R} = (S^1 \times S^2)/Z_2.$$

The exact sequence of homomorphisms is

$$\pi_1(Z_2) \to \pi_1(S^1 \times S^2) \to \pi_1(\tilde{R}) \to \pi_0(Z_2) \to \pi_0(S^1 \times S^2)$$

i.e.

$$0 \to Z \to \pi_1(\tilde{R}) \to Z_2 \to 0.$$

This leads to two possibilities for $\pi_1(\tilde{R})$. Either

$$\pi_1(\tilde{R}) = Z + Z_2$$

or

$$\pi_1(\tilde{R}) = Z.$$

Which is correct can be established by considering the addition law for homotopy group elements. Consider for example two identical elements each represented by a path joining diametrically opposite points in the product space of the circle S^1 with the sphere S^2. The addition of these produces a single circuit of S^1 rather than the identity element. Consequently,

$$\pi_1(\tilde{R}) = Z.$$

The ambiguities arising for the A phase in a magnetic field are resolved in a similar fashion. We have checked all of these results using Mermin's algorithm for computing homotopy groups (Mermin 1978).

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